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IRREDUCIBLE PRODUCTS OF CHARACTERS IN A_n

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ABSTRACT

We show that it is possible to find a pair of non-principal irreducible characters of A_n whose product is irreducible if and only if n is a perfect square.

1. Introduction

Let χ and ψ be non-principal irreducible characters of A_n, $n \geq 5$. Usually the product of these characters is reducible. In this paper we classify the cases in which the product is irreducible. In Theorem 10 we prove that these cases may occur if and only if n is a perfect square. Theorem 10 is proved using some results from the theory of representations of the symmetric group and some previous results of the author concerning products of characters in these groups.

In general one may ask for conditions in which the product of two non-principal characters of a finite simple group is irreducible. No general results are known in this direction. On the other hand, if the group is not simple, then this situation is quite common. For example [I, Corollary 6.17].

2. Basic notation and some results

In this paper we will use the usual notation of character theory that appears, for example in [I]. Fix a positive integer $n \geq 5$. The elements of $\mathrm{Irr}(S_n)$ are indexed by the partitions of n. If $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_h, 0)$ is such a partition then the asociated character is denoted by ζ^α. The permutation character ξ^α is defined

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by $\xi^\alpha = (1_{S_\alpha})^{S_n}$ where S_α denotes the Young subgroup corresponding to α [JK, 2.2.6].

LEMMA 1: *Let G be a finite group and let $H \leq G$. If χ is a class function of G and φ is a class function of H then $(\varphi\chi_H)^G = \varphi^G\chi$. In particular $[\xi^\alpha \cdot \zeta^\beta, \zeta^\gamma] = [\zeta^\beta|_{S_\alpha}, \zeta^\gamma|_{S_\alpha}]$.*

Proof: Let ψ be an arbitrary irreducible character of G. Using Frobenius reciprocity we get:

$$[(\varphi\chi_H)^G, \psi] = [\varphi\chi_H, \psi_H] = [\varphi, (\bar{\chi}\psi)_H] = [\varphi^G, \bar{\chi}\psi] = [\varphi^G\chi, \psi].$$

For every partition α of n and for every positive integer i we define

$$a_i^\alpha = \#\{k | \alpha_k \geq \alpha_{k+1} + i\}.$$

The follwing result is proved in [Z1].

LEMMA 2 ([Z1, Corollary 4.2]): *Let ζ^α be an irreducible character of S_n, $n \geq 5$. Then the follwing holds:*

(i) $[(\zeta^\alpha)^2, \zeta^{(n)}] = 1$.
(ii) $[(\zeta^\alpha)^2, \zeta^{(n-1,1)}] = a_1^\alpha - 1$.
(iii) $[(\zeta^\alpha)^2, \zeta^{(n-2,2)}] = a_1^\alpha \cdot (a_1^\alpha - 2) + a_2^\alpha + a_2^{\alpha'}$.
(iv) $[(\zeta^\alpha)^2, \zeta^{(n-2,1^2)}] = (a_1^\alpha - 1)^2$.

COROLLARY 3 ([Z1, Corollary 4.3]): *If ζ^α is non-linear then*

$$\zeta^{(n-2,2)} \in \mathrm{Irr}\,((\zeta^\alpha)^2).$$

Now we can prove:

THEOREM 4: *The product of two non-linear irreducible characters of S_n is always reducible.*

Proof: Let ζ^α and ζ^β be non-linear elements of $\mathrm{Irr}\,(S_n)$. We have to prove that $[\zeta^\alpha\zeta^\beta, \zeta^\alpha\zeta^\beta] > 1$. But, since the characters of S_n are real valued, this number is equal to $[(\zeta^\alpha)^2, (\zeta^\beta)^2]$. Using Corollary 3 we see that this number is at least 2. ∎

We call an irreducible character of S_n **rectangular** if it is of the form $\zeta^{(m^k)}$.

LEMMA 5:

(i) *If ζ^α is a non-linear and rectangular then $\zeta^{(n-3,1^3)} \in \operatorname{Irr}((\zeta^\alpha)^2)$.*

(ii) *If $\alpha = (m^k)$ and both m and k are larger than 2 then $\zeta^{(n-3,3)} \in \operatorname{Irr}((\zeta^\alpha)^2)$.*

(iii) *If $\alpha = (k+1,k)$ and $k \geq 3$ then $\zeta^{(n-3,3)} \in \operatorname{Irr}((\zeta^\alpha)^2)$.*

Proof: Parts (i) and (ii) follows from [Z2, Lemma 3.2]. We have to prove part (iii). Using the Littlewood–Richardson Rule [JK, 2.8.14] it is possible to evaluate the restriction of $\zeta^{(k+1,k)}$ to the Young subgroup $S_{(n-3,3)}$. The result is:

$$\zeta^{(k+1,k)}|_{S_{(n-3,3)}} = \zeta^{(k+1,k-3)}\#\zeta^{(3)} + \zeta^{(k,k-2)}\#\zeta^{(3)} + \zeta^{(k,k-2)}\#\zeta^{(2,1)} + \zeta^{(k-1,k-1)}\#\zeta^{(2,1)}.$$

Here # denotes the outer tensor product [JK, p. 44]. By Young's Rule [JK, 2.8.2 and 2.8.3]:

$$\xi^{(n-3,3)} = \zeta^{(n)} + \zeta^{(n-1,1)} + \zeta^{(n-2,2)} + \zeta^{(n-3,3)}.$$

Using Lemmas 1 and 2 we get

$$[(\zeta^{(k+1,k)})^2, \zeta^{(n-3,3)}] = [\xi^{(n-3,3)}, (\zeta^{(k+1,k)})^2] - 3$$
$$= [\zeta^{(k+1,k)}|_{S_{(n-3,3)}}, \zeta^{(k+1,k)}|_{S_{(n-3,3)}}] - 3 = 4 - 3 = 1$$

Our proof is therefore complete. ∎

We denote the lexicographic order on the set of partitions of n by $\geq$. We also denote the associated partition with α by α' [JK, P.22].

THEOREM 6: *Suppose that $\alpha \geq \alpha'$ and $\beta \geq \beta'$. The relation $\zeta^\alpha\zeta^\beta = \chi + \psi$ where χ and ψ are distinct elements of $\operatorname{Irr}(S_n)$, holds if and only if (up to order) $\alpha = (n-1,1)$ and β is a non-linear rectangular character.*

Proof: Suppose that $\zeta^\alpha\zeta^\beta = \chi + \psi$ where χ and ψ are distinct elements of $\operatorname{Irr}(S_n)$. This relation implies that $(\zeta^\alpha)^2$ and $(\zeta^\beta)^2$ have exactly two irreducible constituents in common each with multiplicity 1. By Corollary 3 these constituents are $\zeta^{(n)}$ and $\zeta^{(n-2,2)}$. If both ζ^α and ζ^β are non-rectangular then by Lemma 2, $\zeta^{(n-1,1)}$ is a third constituent of both the squares which is not our case. (Note that ζ^α is rectangular if and only if $a_1^\alpha = 1$.) If both ζ^α and ζ^β are rectangular then by Lemma 5 $\zeta^{(n-3,1^3)}$ is a constituent of both $(\zeta^\alpha)^2$ and $(\zeta^\beta)^2$.

We can therefore assume that ζ^α is non-rectangular and ζ^β is a rectangular character. By Lemma 2 we must have $a_1^\alpha \cdot (a_1^\alpha - 2) + a_2^\alpha + a_2^{\alpha'} = 1$. We conclude that $a_1^\alpha = 2$ and $a_2^\alpha + a_2^{\alpha'} = 1$. Since $\alpha \geq \alpha'$ we must have $\alpha = (n-1,1)$ or

$\alpha = (k+1, k)$ where $n = 2k+1$. The latter case is impossible. The reason is as follows. In this case n is odd. By our assumptions $n = km$ is a composite number. It follows that both m and k are greater than 2. By the last two parts of Lemma 5 we conclude that $\zeta^{(n-3,3)}$ is a constituent of both $(\zeta^{\alpha})^2$ and $(\zeta^{\beta})^2$ which is not the case. This completes one direction of the assertion of the theorem.

To prove the other direction note that $1_{S_n} + \zeta^{(n-1,1)} = \xi^{(n-1,1)}$. Therefore

$$\zeta^{(n-1,1)} \cdot \zeta^{(m^k)} = \xi^{(n-1,1)} \cdot \zeta^{(m^k)} - \zeta^{(m^k)}.$$

By the Branching Theorem [JK, 2.4.3] and Lemma 1 the result is $\zeta^{(m+1,m^{k-2},m-1)} + \zeta^{(m^{k-1},m-1,1)}$. This proves our theorem. ∎

3. Proof of the main result

As in [Z1] and [Z2] we devide the irreducible characters of A_n into two types. The first consists of characters of the form $\zeta^{\alpha}|_{A_n}$ where $\alpha > \alpha'$. We call these characters **characters of type A**. If $\alpha = \alpha'$ then the restriction of ζ^{α} to A_n is a sum of two distinct irreducible characters of A_n. These characters are **characters of type B**.

Application of Theorem 4 to characters of type A yields:

LEMMA 6: *The product of two non-principal characters of A_n of type A is always reducible.*

Let χ be a non-principal character of A_n of type A and let φ be a character of type B. Suppose that $\zeta^{\alpha}|_{A_n} = \chi$ and that $\varphi^{S_n} = \zeta^{\beta}$. Suppose further that $\chi\varphi$ is irreducible. In this case by Lemma 1 we have: $\zeta^{\alpha} \cdot \zeta^{\beta} = (\chi \cdot \psi)^{S_n}$. Induction of an irreducible character of A_n to S_n is either irreducible or the sum of two distinct elements of $\mathrm{Irr}(S_n)$. The first possibility does not occur by Theorem 4. Therefore we can use Theorem 6 and conclude that $\beta = \beta'$ is rectangular. This may occur only if n is a perfect square. On the other hand if $n = k^2$ then by the proof of Theorem 6 we know that

$$\zeta^{(n-1,1)} \cdot \zeta^{(k^k)} = \zeta^{(k+1,k^{k-2},k-1)} + \zeta^{(k^{k-1},k-1,1)}.$$

Write $\chi = \zeta^{(n-1,1)}|_{A_n}$ and $\psi + \psi' = \zeta^{(k^k)}|_{A_n}$. Then $\chi\psi = \zeta^{(k^{k-1},k-1,1)}|_{A_n} \in \mathrm{Irr}(A_n)$. Combining all of these we prove:

LEMMA 7: *We can find $1_{A_n} \neq \chi \in \mathrm{Irr}(A_n)$ of type A and $\psi \in \mathrm{Irr}(A_n)$ of type B with $\chi\psi \in \mathrm{Irr}(A_n)$ if and only if n is a perfect square.*

Now we have to deal with the case in which both characters are of type B. We begin with a simple Lemma.

LEMMA 8: *Suppose that $\alpha = \alpha'$ and $\beta = \beta'$ are partitions of n. If $n \geq 5$ is not a square then $[(\zeta^\alpha)^2, (\zeta^\beta)^2] > 8$.*

Proof: By the conditions on α we must have $a_1^\alpha \geq 2$ and $a_2^\alpha = a_2^{\alpha'} \geq 1$. Therefore by Lemma 2 $[(\zeta^\alpha)^2, \zeta^{(n-2,2)}] \geq 2$. Since $\alpha = \alpha'$ we also have $[(\zeta^\alpha)^2, \zeta^{(2^2,1^{n-4})}] \geq 2$. The same relations hold for ζ^β. Since both these characters are also real valued, the Lemma follows. ∎

LEMMA 9: *If $n \geq 5$ is not a square and $\chi, \psi \in \mathrm{Irr}(A_n)$ are of type B then $\chi\psi$ is reducible.*

Proof: Let $\alpha = \alpha'$, $\beta = \beta' \neq \alpha$, $\zeta^\alpha|_{A_n} = \chi + \chi'$ and $\zeta^\beta|_{A_n} = \psi + \psi'$. If $\chi\psi$ is irreducible then $\chi\psi'$ is also irreducible. Therefore by Lemma 1

$$[\zeta^\alpha \cdot \zeta^\beta, \zeta^\alpha \cdot \zeta^\beta] = [\chi^{S_n}\zeta^\beta, \chi^{S_n}\zeta^\beta] = [(\chi\psi + \chi\psi')^{S_n}, (\chi\psi + \chi\psi')^{S_n}] \leq 8.$$

However this is impossible in light of Lemma 8.

To complete the proof we have to prove that χ^2 and $\chi\chi'$ are reducible. χ^2 is reducible by [Z2]. If χ is not real valued then $\chi\chi' = \chi\bar{\chi}$ and this last character is reducible. On the other hand if χ is real valued we can use [Z1, Lemma 5.5] to show that $\zeta^{(n-2,2)}|_{A_n}$ is a constituent of both χ^2 and $(\chi')^2$. Therefore $[\chi\chi', \chi\chi'] = [\chi^2, (\chi')^2] \geq 2$. This complets the proof of the Lemma. ∎

Using Lemmas 6,7 and 9 we can prove our main Theorem

THEOREM 10: *Let ≥ 5. There are non-principal $\chi, \psi \in \mathrm{Irr}(A_n)$ with $\chi\psi \in \mathrm{Irr}(A_n)$ if and only if n is a perfect square.*

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